Measure and Integration - Final exam solutions

1) Necessarily true: (b), (d), (e)

2) a) • $\Omega \in \mathcal{A}$ since $\Omega^c = \phi$ which is countable

• $A \in \mathcal{A} \Rightarrow \underset{\underset{(A^c)^c}{\parallel}}{A}$ countable or A^c countable $\Rightarrow A^c \in \mathcal{A}$

• Let $A_n \in \mathcal{A}$, $n \in \mathbb{N}$. If all the A_n's are countable then $\bigcup_{n=1}^{\infty} A_n$ is countable, so $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$.

Otherwise there exists N such that $(A_N)^c$ is countable; then, $\left(\bigcup_{n=1}^{\infty} A_n\right)^c = \bigcap_{n=1}^{\infty} A_n^c \subset A_N^c$, so $\left(\bigcup_{n=1}^{\infty} A_n\right)^c \in \mathcal{A}$, so $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$.

b) Each set $\{x\}$ is countable, so

$$\{\{x\} : x \in \Omega\} \subset \mathcal{A}$$

$$\Rightarrow \sigma(\{\{x\} : x \in \Omega\}) \subset \sigma(\mathcal{A}) \overset{\mathcal{A} \text{ is a } \sigma\text{-algebra}}{=} \mathcal{A}$$

Now fix $A \in \mathcal{A}$. If A is countable, then

$$A = \bigcup_{x \in A} \{x\} \in \sigma(\{\{x\} : x \in \Omega\})$$

If A^c is countable, then

$$A = \left(\bigcup_{x \in A^c} \{x\} \right)^c \in \sigma(\{\{x\} : x \in \Omega\})$$

Hence, $\mathcal{A} \subset \sigma(\{\{x\} : x \in \Omega\})$.

3) a) Let Ω be a set. A function $\mu^* : P(\Omega) \to [0, \infty]$ is an outer measure if (1) $\mu^*(\phi) = 0$

(2) $A \subset B \Rightarrow \mu^*(A) \leq \mu^*(B)$

(3) $A_n \subset \Omega, n \in \mathbb{N} \Rightarrow \mu^*\left(\bigcup^{\infty} A_n\right) \leq \sum^{\infty} \mu^*(A_n)$

b) For all $x \in \mathbb{R}$, the set $\{x\}$ is closed, hence Borel measurable.

Moreover, $m(\{x\}) = \lim_{n \to \infty} m\left([x - \frac{1}{n}, x + \frac{1}{n}]\right) = \lim_{n \to \infty} \frac{2}{n} = 0$.

4) a) Let $\mathcal{F} = \{A \subset \Omega_2 : f^{-1}(A) \in \mathcal{A}_1\}$. $\mathcal{F}$ is a σ-algebra:

- $\Omega_2 \in \mathcal{F}$ since $f^{-1}(\Omega_2) = \Omega_1 \in \mathcal{A}_1$,
- if $A \in \mathcal{F}$, then $f^{-1}(A^c) = (f^{-1}(A))^c \in \mathcal{A}_1$, so $A^c \in \mathcal{F}$,
- if $A_n \in \mathcal{F}, n \in \mathbb{N}$, then $f^{-1}\left(\bigcup A_n\right) = \bigcup f^{-1}(A_n) \in \mathcal{A}_1$, so $\bigcup A_n \in \mathcal{F}$.

Moreover, $\mathcal{E} \subset \mathcal{F}$. Hence, $\mathcal{A}_2 = \sigma(\mathcal{E}) \subset \sigma(\mathcal{F}) = \mathcal{F}$.

This means: $\forall B \in \mathcal{A}_2$ we have $f^{-1}(B) \in \mathcal{A}_1$. So f is measurable.

b) Let $g_n(x) = n \cdot (f(x+\frac{1}{n}) - f(x))$. Then, g_n is measurable and $g_n \xrightarrow{n\to\infty} f'$, so f' is measurable.

5) a) $v(\phi) = \int_\phi f \, d\mu = \int_\Omega f \cdot \mathbb{1}_\phi \, d\mu = \int_\Omega 0 \, d\mu = 0$.

$A_n \in \mathcal{A}$ pairwise disjoint $\Rightarrow$

$$v(\bigcup^\infty A_n) = \int_{\cup A_n} f \, d\mu = \int_\Omega f \cdot \mathbb{1}_{\cup A_n} \, d\mu =$$

$$= \int_\Omega f \sum \mathbb{1}_{A_n} \, d\mu \overset{(*)}{=} \sum \int_{A_n} f \, d\mu = v(A_n);$$

the equality (*) holds because the functions $f \cdot \mathbb{1}_{A_n}$ are all non-negative.

b) $f_n \to f \Rightarrow f_n^p \to f^p$, and the convergence is dominated:

$|f_n|^p \leq g^p$ and $\int |g|^p \, d\mu < \infty$ since $g \in \mathcal{L}^p(\Omega)$

Hence, by the Dominated Convergence Theorem, $\int |f|^p \, d\mu < \infty$, so $f \in \mathcal{L}^p(\Omega)$.

We also have $|f_n - f|^p \longrightarrow 0$ and

$$|f_n - f|^p \leq (|f_n| + |f|)^p \leq 2^p \, g^p \to \text{integrable},$$

so, again by the Dominated Convergence Theorem,

$$\|f_n - f\|_p = \left(\int |f_n - f|^p \, d\mu\right)^{1/p} \longrightarrow 0.$$

6) Since the measure spaces

$(\Omega_1, \mathcal{A}_1, \mu)$, $(\Omega_1, \mathcal{A}_1, \mu')$, $(\Omega_2, \mathcal{A}_2, \nu)$, $(\Omega_2, \mathcal{A}_2, \nu')$

are all finite, hence σ-finite, we have, $\forall D \in \mathcal{A}_1 \otimes \mathcal{A}_2$,

$$(\mu \otimes \nu)(D) = \int_{\Omega_1} \nu(D_{\omega_1})\, d\mu(\omega_1), \qquad (*)$$

$$(\mu' \otimes \nu')(D) = \int_{\Omega_1} \nu'(D_{\omega_1})\, d\mu'(\omega_1).$$

If $(\mu \otimes \nu)(D) = 0$, then the right-hand side of $(*)$ is 0. Since the function $\omega_1 \mapsto \nu(D_{\omega_1})$ is nonnegative with integral $=0$, this function $=0$ almost everywhere. That is, $\exists E \subset \Omega_1$: $\mu(E) = 0$ and $\nu(D_{\omega_1}) = 0 \ \forall \omega_1 \in E^c$. By the assumptions, we obtain: $\mu'(E) = 0$ and $\nu'(D_{\omega_1}) = 0 \ \forall \omega_1 \in E^c$. Hence,

$$(\mu' \otimes \nu')(D) = \int_{\Omega_1} \nu'(D_{\omega_1})\, d\mu'(\omega_1)$$

$$= \int_{E^c} \nu'(D_{\omega_1})\, d\mu'(\omega_1) = 0.$$